The almost Gorenstein Rees algebras of socle ideals

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§1 History of almost Gorenstein rings

- [Barucci–Fröberg, 1997]
  $\cdots$ one–dimensional analytically unramified local rings
- [Goto–Matsuoka–Phuong, 2013]
  $\cdots$ one–dimensional Cohen–Macaulay local rings
- [Goto–Takahashi–T, 2015]
  $\cdots$ higer dimensional Cohen–Macaulay local/graded rings

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- Characterized the almost Gorenstein property of $\mathcal{R}(I)$ where I is
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§2 Definition of almost Gorenstein rings

Setting 2.1 (local case)

- $(R, \mathfrak{m})$ a Cohen-Macaulay local ring with $d = \dim R$.
- $\exists$ the canonical module K_R.
- $|R/\mathfrak{m}| = \infty$.

Definition 2.2

We say that R is *an almost Gorenstein local ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}_{\mathfrak{m}}^0(C)$.

Therefore in Definition 2.2, if $C \neq (0)$, then C is Cohen–Macaulay and $\dim_R C = d - 1$. Moreover

$$\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C) \Longleftrightarrow \mathfrak{m}C = (f_2, f_3, \ldots, f_d)C$$

for some $f_2, f_3, \ldots, f_d \in \mathfrak{m}$.

Hence C is a *maximally generated Cohen–Macaulay module* in the sense of B. Ulrich (cf. [2]), which is called *an Ulrich R-module*.

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Example 2.3

(1) $k[[t^3, t^4, t^5]]$.

(2) $k[[t^a, t^{a+1}, \ldots, t^{2a-3}, t^{2a-1}]]$ $(a \geq 4)$.

(3) $k[[X, Y, Z]]/(X, Y) \cap (Y, Z) \cap (Z, X)$.

(4) Suppose that R is not Gorenstein. If R is an almost Gorenstein local ring, then R is G-regular.

(5) 1–dimensional finite CM–representation type.

(6) 2–dimensional rational singularity.

Setting 2.4 (graded case)

- $R = \bigoplus_{n \geq 0} R_n$ a Cohen–Macaulay graded ring with $d = \dim R$
- $(R_0, \mathfrak{m})$ a local ring
- $\exists$ the graded canonical module K_R
- $\mathfrak{M} = \mathfrak{m}R + R_+$
- $a = \mathrm{a}(R) := -\min\{n \in \mathbb{Z} \mid [\mathrm{K}_R]_n \neq (0)\}$
- $|R/\mathfrak{m}| = \infty$

Definition 2.5

We say that R is *an almost Gorenstein graded ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$

of graded R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$.

Notice that

- R is an almost Gorenstein graded ring
  $\Longrightarrow R_{\mathfrak{M}}$ is an almost Gorenstein local ring.

Theorem 2.6 ([Goto–Takahashi–T, 2015])

Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 3$ and Q a parameter ideal of R. Then TFAE.

(1) $\mathcal{R}(Q)$ *is an almost Gorenstein graded ring.*

(2) $Q = \mathfrak{m}$.

§3 Main results

In this section

- $(R, \mathfrak{m})$ a Gorenstein local ring with $d = \dim R \geq 3$
- $|R/\mathfrak{m}| = \infty$
- I an $\mathfrak{m}$-primary ideal in R which contains a parameter ideal Q as a reduction (i.e., $\exists\ r \geq 0$ s.t. $I^{r+1} = QI^r$)
- $J := Q : I$
- $\mathcal{R} = \mathcal{R}(I) := R[It] \subseteq R[t]$ the Rees algebra of I
- $\mathfrak{M} := \mathfrak{m}\mathcal{R} + \mathcal{R}_+$ a unique graded maximal ideal of $\mathcal{R}$

Fact 3.1

Suppose that $I^2 = QI$. *Then*

- [Goto–Shimoda, 1979] $\mathcal{R}$ *is a Cohen–Macaulay ring.*
- [Ulrich, 1996] *One has*

$$\mathrm{K}_{\mathcal{R}}(1) \cong \sum_{i=0}^{d-3} \mathcal{R}{\cdot}t^i + \mathcal{R}{\cdot}Jt^{d-2}$$

as a graded $\mathcal{R}$*–module.*

Note: $\mathrm{a}(\mathcal{R}) = -1$ and $\dim \mathcal{R} = d+1$.

Let $\mathrm{r}(\mathcal{R})$ denote the Cohen-Macaulay type of $\mathcal{R}$.

Corollary 3.2

Suppose that $I^2 = QI$. Then $\mathrm{r}(\mathcal{R}) = d - 2 + \mu_R(J/I)$. In particular, $\mathcal{R}$ is a Gorenstein ring if and only if $I = J$ and $d = 3$.

Proposition 3.3

Let Q be a parameter ideal of R such that $Q \subseteq \mathfrak{m}^2$. Set $I = Q : \mathfrak{m}$. Then $\mathcal{R}_{\mathfrak{M}}$ is NOT an almost Gorenstein local ring.

Proposition 3.4

Suppose that $d = 3$. Let Q be a parameter ideal of R and set $I = Q : \mathfrak{m}$. If $I^2 = QI$ and $\mathfrak{m}^2 \subseteq I$, then $\mathcal{R}$ is an almost Gorenstein graded ring.

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Proposition 3.4

Suppose that $d = 3$. Let Q be a parameter ideal of R and set $I = Q : \mathfrak{m}$. If $I^2 = QI$ and $\mathfrak{m}^2 \subseteq I$, then $\mathcal{R}$ is an almost Gorenstein graded ring.

The main result of this talk is stated as follows.

Theorem 3.5

Suppose that R is a RLR. Let Q be a parameter ideal of R such that $Q \neq \mathfrak{m}$. Set $I = Q : \mathfrak{m}$. Then TFAE.

(1) *$\mathcal{R}(I)$ is an almost Gorenstein graded ring.*

(2) *Either $I = \mathfrak{m}$, or $d = 3$ and $I = (x) + \mathfrak{m}^2$ for $\exists\ x \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

The end of this talk, let us note one example.

Example 3.6

Let $R = k[[x, y, z]]$ be the formal power series ring over an infinite field k. We set $\mathfrak{m} = (x, y, z)$, $Q = (x, y^2, z^n)$ with $n \geq 2$, and $I = Q : \mathfrak{m}$. Then $I = (x, y^2, yz^{n-1}, z^n)$ and $I^2 = QI$.

(1) If $n = 2$, then $I = (x) + \mathfrak{m}^2$, so that $\mathcal{R}(I)$ is an almost Gorenstein graded ring.

(2) Suppose $n \geq 3$. Then $I \neq \mathfrak{m}$, $Q \neq \mathfrak{m}$, and $I \neq (f) + \mathfrak{m}^2$ for any $f \in \mathfrak{m} \setminus \mathfrak{m}^2$. Hence $\mathcal{R}(I)$ is NOT an almost Gorenstein graded ring.

Thank you so much for your attention.

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